Amplitude and Phase Modulation

I believe I am not the only one to struggle with the spectrum of a phase modulated signal. Even the pure tone phase modulation spectrum consists of an infinite number of sidebands whose phase changes from one sideband to the next one and whose amplitudes are given by the Bessel functions.

This document presents a purely pictorial representation of amplitude and phase modulation.

Acknowledgments

Many thanks to Nikola Bucalovic and Andrin Doll for useful discussions. Dedicated to my teacher Bart van Wees for explaining physics with pictures.

1.1 Background

I started writing this note while working for Zurich Instruments, a Swiss maker of scientific instrumentation. At that time, the company was selling only one product, a lock-in amplifier, and we were naturally tackling all customers' problems in terms of demodulation.

In short, a lock-in amplifier finds the amplitude A and phase φ of an input signal of the type

$$V_S(t) = A\cos(2\pi f_C t + \varphi). \tag{1.1}$$

by a process called *demodulation*. The existing literature explains it with the multiplication of the input signal by sine and cosine of the reference phase, also referred to as the in-phase and quadrature of the reference but I found this explanation lacked in clarity.

The multiplication by sine and cosine is equivalent to looking at the signal from a reference frame rotating with the reference phase. This element is key for the pictorial representation of amplitude and phase modulation.

1.2 Complex Phasor to Represent Real Signals

The signal of eq. (1.1) can also be written in the complex plane as the sum of two phasors, each one of length $\frac{1}{2}A$ rotating at the same speed $\omega_C = 2\pi f_C$, in opposite directions:

$$\frac{A}{2}\left[e^{j(\omega_C t+\varphi)} + e^{-j(\omega_C t+\varphi)}\right].$$

From a spectral point of view, the two phasors are the exact copy of each other, one at positive frequency $+f_C$ and the other at negative frequency $-f_C$, with amplitude $\frac{1}{2}A$ and phase φ.

1.3 Signal Modulation

A generic signal can be written as

$$A(t)\cos[\omega_C t + \varphi(t)] = \frac{1}{2}A(t)\left[e^{j[\omega_C t+\varphi(t)]} + e^{-j[\omega_C t+\varphi(t)]}\right] \tag{1.2}$$

with a constant carrier angular frequency ω_C and adequate $A(t)$ and $\varphi(t)$.

We discard the negative frequency component $-f_C$ term because it has the same spectral content as the one at positive frequency, just mirrored. We also neglect the constant factor $\frac{1}{2}$ that is irrelevant for amplitude and phase modulation:

$$A(t)e^{j[\omega_C t+\varphi(t)]}.$$

Next, we want to get rid of the carrier, which only shifts the spectrum by the constant quantity ω_C but carries otherwise no useful information. To do so, we multiply the quantity above by $e^{-j\omega_C t}$.

The multiplication by $e^{-j\omega_C t}$ can also be interpreted as a change of the reference frame: indeed complex numbers provide a convenient way of dealing with rotations on a 2D plan.

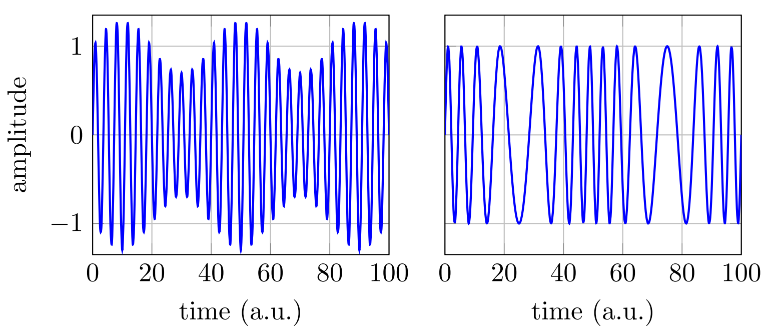

Figure 1.1: Examples of pure amplitude modulated (left) and pure phase modulated (right) signals: in the first case the amplitude is time-dependent, in the second the spacing between successive zero crossings is time-dependent, while the amplitude remains constant.

Suppose to be standing at the origin and rotate anti-clockwise with angular velocity ω_C. In this *rotating frame*[1], the phasor will appear as essentially steady[2] except for the amplitude and phase modulation given by

$$d(t) = A(t)e^{j\varphi(t)} \tag{1.3}$$

In general this signal is neither a pure amplitude nor a frequency modulated signal since both vary as shown by the black arrow in Fig. 1.2. In an amplitude modulated signal, only the phasor length $A(t)$ changes (the blue line in the figure) but not its phase: in other words, all the changes are along the phasor direction and not perpendicular to it. Phase modulation on the other hand would instead move the phasor on a circle (red line), with no changes to its radius.

1.4 Amplitude Modulation

We will see that the sidebands of amplitude and (narrow band) phase modulations differ only by an angle: that is why they are indistinguishable by Fourier power spectral analysis because the phase information is lost when squaring the Fourier transform. It is easy to understand the origin of the phase difference using complex phasors.

[1]Note that I cannot choose the initial orientation of the rotating frame to have the phase φ_C, that is by multiplying the signal by $e^{-j[\omega_C t+\varphi_C]}$ because the negative component in this reference frame would have an initial phase of $-2\varphi_c$, thereby having a different phase compared to the positive frequency.

[2]The change of reference frame is a common trick to remove the large fast dynamics that hides the small slow but interesting one.

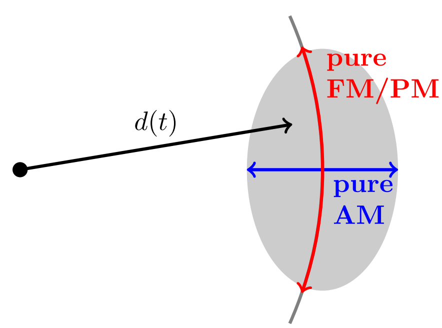

Figure 1.2: The phasor of a generically modulated signal can sweep the grey area. In pure AM, only the length $A(t)$ is time dependent, in pure phase modulation, only the phase changes. The amplitude and phase modulations are *not* restricted to be a fraction of the radius and π respectively.

A general AM signal has a modulated amplitude $A(t)$ and a constant phase offset $\varphi(t)$. To be more specific, we take a pure tone AM signal

$$d_{\text{AM}}(t) \equiv 1 + h\sin(\omega_m t) = 1 - \frac{jh}{2}\left[e^{+j\omega_m t} - e^{-j\omega_m t}\right] \tag{1.4}$$

with average amplitude 1, amplitude modulation h, angular frequency ω_m (typically $\omega_m \ll \omega_C$ although this is not relevant for the discussion) and constant phase offset $\varphi(t)$ that I will for the moment take equal to zero.

We need to consider both modulation phasors of eq. (1.4). With reference to Fig. 1.3 a), the modulation of the amplitude, $h\sin(\omega_m t)$, originates from the two phasors of length $\frac{h}{2}$ rotating in opposite directions, each with frequency $|f_m|$ but with their sum, the thick blue arrow, always along the x–axis. These are the two sidebands of the AM spectrum.

1.4.1 Phase of AM Sidebands

What about their relative phase in the Fourier transform? The pictorial representation provides the answer. The phase is determined by the phasors' value at $t = 0$: phasor pointing along the x–axis have real amplitude, phasors pointing along y imaginary amplitude.

In the case of AM, the signal is only modulated in the direction collinear with the carrier phasor and the two sidebands must be at an angle such that their sum is the same as that of the carrier.

For instance, for the modulation of eq. (1.4), the phasor with negative frequency has value $+\frac{jh}{2}$ at $t = 0$ or $+90^\circ$ with respect to the carrier; the other at positive frequency has value $-\frac{jh}{2}$ or $+90^\circ$ to the carrier.

Had we instead chosen an amplitude modulation of the form

$$1 + h\cos(\omega_m t) = 1 + \frac{h}{2}\left[e^{+j\omega_m t} + e^{-j\omega_m t}\right] \tag{1.5}$$

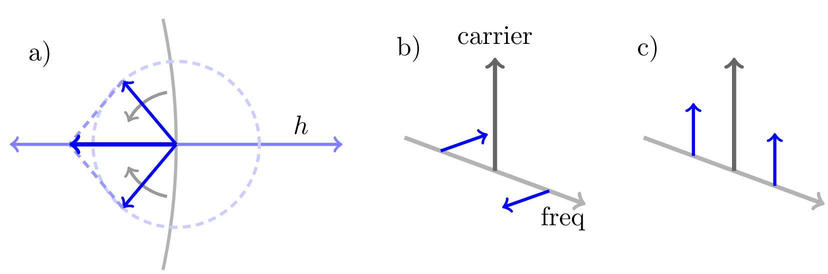

Figure 1.3: a) The decomposition of the sidebands of an AM signal: the sidebands must add up to have only a contribution parallel to the carrier phasor. b) AM for modulation of the type $\sin(\omega_m t)$ and c) $\cos(\omega_m t)$. In the drawing we have assumed the carrier phase offset to be zero such that the carrier phasor is pointing in the x–direction.

the two modulation phasors at $t = 0$ would have both been pointing along the $x-$ axis, collinear with the carrier phasor: their Fourier transform would have given a spectrum with all phasors having the same relative phase as in Fig. 1.3 c).

1.5 Phase Modulation

In an PM signal, the situation is more complicated. We refer again to eq. (1.3): a general PM signal has a modulated phase $\varphi(t)$ and a constant amplitude $A(t)$ assumed here for simplicity to be $A(t) = 1$.

Consider now a pure tone PM signal

$$d_{\mathrm{PM}}(t) \equiv e^{jh\sin(\omega_m t)} \tag{1.6}$$

with modulation index h. Is there an intuitive way of showing why the PM signal of eq. (1.6) has a frequency spectrum consisting of an infinite number of sidebands at $\pm n\omega_m$ and the amplitude of the n-th sideband is given by the n-th Bessel function?

1.5.1 Pictorial Representation of First Set of Sidebands

$d_{\mathrm{PM}}(t)$ has constant amplitude, so it describes a phasor that moves on a circle of radius 1. For a small phase swing, $h \ll 1$, we can neglect the curvature of the circle, see Fig. 1.4 a) and two terms only are sufficient to approximate $d_{\mathrm{PM}}(t)$, one real, constant, b_0 and one imaginary, that reverses direction every time $\varphi_{\mathrm{PM}}(t)$ does so, so it must be, to first approximation, of the form $b_1 \sin(\omega_m t)$.

To find the approximated values of b_0 and b_1, we impose that $e^{jh\sin(\omega_m t)}$ and its approximation

$$b_0 + jb_1 \sin(\omega_m t)$$

are equal at some values of $\omega_m t$, for instance that the real parts are equal at $\omega_m t = 0$ and the imaginary parts at $\omega_m t = \frac{\pi}{2}$. This choice gives

$$b_0(h) = 1, \quad b_1(h) = \sin h \approx h.$$

A different choice of points would have given a slightly different approximation[3].

This small h regime, typically $h < 0.2$, is called *narrow band* PM because it can be fully described by a carrier and the two sidebands b_1 at $\pm\omega_m$, thereby occupying a very limited portion of the frequency spectrum.

1.5.2 Phase of FM Sidebands

The small h approximation of the phase modulation

$$1 + jh\sin(\omega_m t) = 1 + \frac{h}{2}\left[e^{+j\omega_m t} - e^{-j\omega_m t}\right] \tag{1.7}$$

[3] An alternative way to find the values of b_0 and b_1 makes use of the small argument approximation (*i.e.* Taylor expansion) for the exponential:

$$e^{jh\sin(\omega_m t)} \approx 1 + jh\sin(\omega_m t) \;\rightarrow\; b_0 = 1,\; b_1(h) = h.$$

This is consistent with the previous result since for small h we have $\sin h \approx h$.

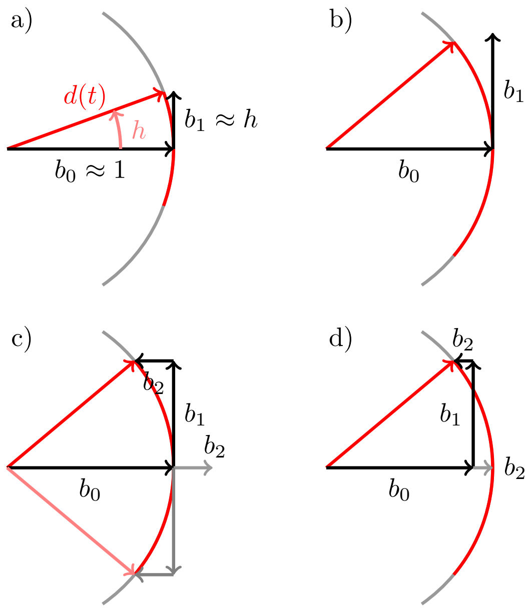

Figure 1.4: The PM signal and its decomposition in terms of the Bessel functions. a) In the small modulation approximation $h \ll 1$, one can neglect the curvature of the circle and two terms only, b_0 and $b_1 \approx h$, give a good approximation. b) When the modulation index h is large, the approximation breaks down. c) An extra term b_2 is needed to obtain a better approximation. This term must contribute twice per cycle of the modulation phasor and must therefore have a $\cos(2\omega_m t)$ dependence. d) $\tilde{b}_2(t) = b_2 \cos(2\omega_m t)$ gives a positive contribution at $t = 0$: in order for their sum to be 1, b_0 must decrease.

looks very similar to the amplitude modulation eq. (1.4), two sidebands at $\pm f_m$. In the PM case, however, the signal is modulated in the direction *orthogonal* to the carrier as Fig. 1.5 a) shows and the two sidebands must be at an angle such that their sum is 180° with the carrier.

For the modulation of eq. (1.7), the sideband at positive frequency $+f_m$ has amplitude $+\frac{h}{2}$ at $t = 0$ and the sideband at negative frequency $-\frac{h}{2}$, as shown in Fig. 1.5 b).

Had we instead chosen a phase modulation of the form

$$e^{jh\cos(\omega_m t)} \approx 1 + jh\cos(\omega_m t) = 1 + \frac{jh}{2}\left[e^{+j\omega_m t} - e^{-j\omega_m t}\right]$$

with the two modulation phasors at $t = 0$ having the purely imaginary amplitude $\frac{jh}{2}$, the Fourier transform would have looked like Fig. 1.5 c).

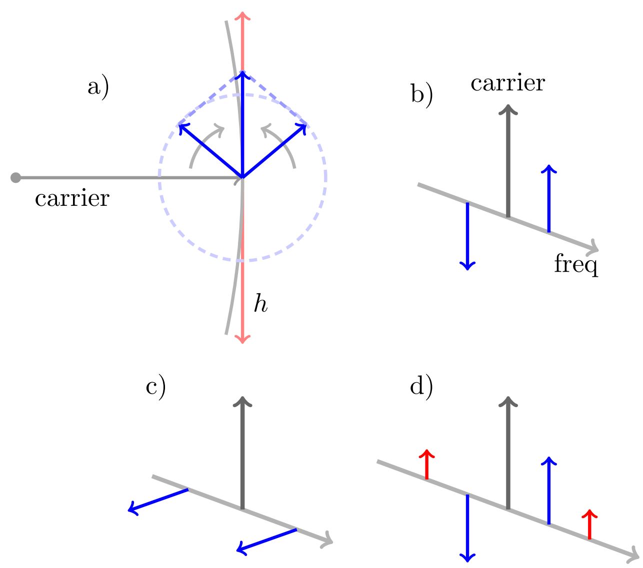

Figure 1.5: a) The decomposition of the sidebands of a small modulation index PM signal: the sidebands must add up to have only a contribution parallel to the carrier phasor. b) Fourier transform of a PM signal with time dependent modulation $\sin(\omega_m t)$ and c) $\cos(\omega_m t)$. d) Fourier spectrum including the $2f_m$ sidebands (red).

1.6 Pictorial representation of Higher Order Sidebands

In the case of large phase modulation (large h), the approximation above fails to describe a phasor with unity amplitude, the approximation being the worst when $d_{\mathrm{PM}}(t)$ reaches its top-most and its bottom-most points, see Fig. 1.4 b).

We need to compensate it with an additional term of amplitude $b_2(h)$ in the collinear direction, Fig. 1.4 c): such correction must be a negative quantity twice per cycle of the phasor $d_{\mathrm{PM}}(t)$, at its top-most and bottom-most points, so it must have a $\cos(2\omega_m t)$ dependence and therefore it accounts for the second set of sidebands at $\pm 2\omega_m$.

The new term

$$\bar{b}_2(t) = b_2 \cos(2\omega_m t)$$

gives also a positive contribution at $t = 0$. If we want the modulation phasor at $t = 0$ to have unit length, b_0 needs to decrease, as in Fig. 1.4 d). Therefore $b_0(h)$ and $b_2(h)$ must satisfy

$$\begin{cases} b_0 + b_2 = 1 & \text{when } d_{\mathrm{PM}}(t) \text{ is horizontal} \\ b_0 - b_2 = \cos h & \text{when } d_{\mathrm{PM}}(t) \text{ reaches its extremal point, } \omega_m t = \frac{\pi}{2} \end{cases}$$

which gives

$$b_0(h) = \frac{1+\cos h}{2} \qquad b_2(h) = \frac{1-\cos h}{2}. \tag{1.8}$$

At this point, we still do not know we are dealing with the Bessel functions $J_n(h)$, but also $J_0(h)$ has one maximum (its global) at $h = 0$ and $J_1(h)$ is linear in h for small h.

$\bar{b}_2(t)$, being purely real, contributes to the approximation as if it was an amplitude modulation. Similar to the discussion for eq. (1.5), the two modulation phasors at $t = 0$ are collinear with the carrier phasor and the Fourier spectrum appears as depicted in Fig. 1.5 d).

1.6.1 Large Phase Modulation

The pictorial representation also helps us to conclude that

1. the perpendicular component must be an odd function of time and the collinear an even one[4];

2. only $\sin(3\omega_m t)$, $\sin(5\omega_m t)\ldots$ contribute to the vertical component and only $\cos(4\omega_m t)$, $\cos(6\omega_m t)\ldots$ to the horizontal one. Indeed the phasor $e^{jh\sin(\omega_m t)}$ reaches its turning point at $\frac{\pi}{2}$ and traces its path back, a symmetry that the terms $\sin(2\omega_m t)$ and $\cos(3\omega_m t)$ do not possess. We thus write $d_{\mathrm{PM}}(t)$ as

$$\begin{aligned} e^{jh\sin(\omega_m t)]} = &\sum_{n\geq 0} b_{2n}(h)\cos[2n\omega_m t] \\ &+ j\sum_{n\geq 0} b_{2n+1}(h)\sin[(2n+1)\omega_m t]; \end{aligned} \tag{1.9}$$

3. In the Fourier spectrum, all sine sidebands behave similar to $\sin(\omega_m t)$ and all cosine sidebands similar to $\cos(2\omega_m t)$.

Only a few terms are sufficient to give a fairly good approximation. Fig. 1.6 shows this with a modulation index $h = 0.7\pi$ sufficiently large that the vertical component of $d_{\mathrm{PM}}(t)$ decreses around $\omega_m t = \frac{\pi}{2}$. This behavior cannot be described by the term $\sin(\omega_m t)$ that only increases monotonically in $0 \leq \omega_m t < \frac{\pi}{2}$ and the third sideband

$$\bar{b}_3(t) = b_3\sin(3\omega_m t)$$

accounts for a negative contribution necessary for the correction, as shown in Fig. 1.6 c).

To find the values of the two coefficients b_1 and b_3, we impose that our approximation equals $d_{\mathrm{PM}}(t)$ at three arbitrary times: $\omega_m t = 0$ (only the real

[4]One can reach the same conclusion by looking at the definition

$$e^{jh\sin(\omega_m t)} = \cos[h\sin(\omega_m t)] + j\sin[h\sin(\omega_m t)].$$

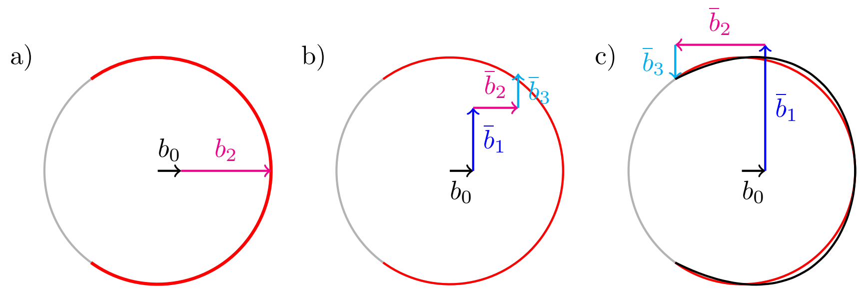

Figure 1.6: Phasor representing a PM signal with $h = 0.7\pi$ for different phases $\omega_m t$ and its approximation. a) $\omega_m t = 0$ (the terms $\bar{b}_1$ and $\bar{b}_3$ are not plot); b) $\omega_m t = \frac{\pi}{6}$ and c) $\omega_m t = \frac{\pi}{2}$. Note the term $\bar{b}_3$ giving a negative contribution to account for the phase modulation when $h > \frac{\pi}{2}$. The black trace represents the path traced by this approximation.

part), $\omega_m t = \frac{\pi}{4}$ (only the imaginary part), and $\omega_m t = \frac{\pi}{2}$ (both real and imaginary components). With our choice, we find that

$$b_1(h) = \frac{\sqrt{2}\sin\frac{h}{\sqrt{2}} + \sin h}{2}, \quad b_3(h) = \frac{\sqrt{2}\sin\frac{h}{\sqrt{2}} - \sin h}{2}; \tag{1.10}$$

b_0 and b_2 are still given by eq. (1.8).

1.7 The Relation with the Bessel Coefficients

We want now to manipulate eq. (1.9) and show that $b_n(h)$ are proportional to the Bessel coefficients.

To show this, let us use $\tau \equiv \omega_m t$ and rewrite eq. (1.9):

$$\begin{aligned} e^{jh\sin\tau} = b_0(h) &+ \sum_{n\geq 1} \frac{b_{2n}(h)}{2}\left[e^{2jn\tau} + e^{-2jn\tau}\right] \\ &+ j\sum_{n\geq 0} \frac{b_{2n+1}(h)}{2j}\left[e^{(2n+1)j\tau} - e^{-(2n+1)j\tau}\right] \end{aligned}$$

where in the first line, I have separated the constant term from the time dependent ones in preparation for further manipulation. I now *define*

$$J_m(h) \equiv \begin{cases} b_0(h) & \text{if } m = 0 \\ \frac{1}{2}b_m(h) & \text{if } m \text{ is positive or even negative} \\ -\frac{1}{2}b_m(h) & \text{if } m \text{ is odd negative} \end{cases} \tag{1.11}$$

to obtain

$$e^{jh\sin\tau} = \sum_{n=-\infty}^{+\infty} J_n(h) e^{jn\tau}$$

the standard generating expression for the Bessel functions.

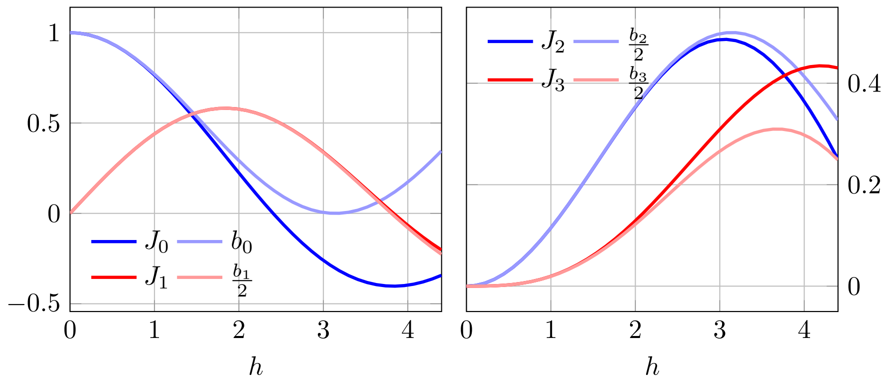

Figure 1.7: Comparison between the first Bessel functions and their approximations eq. (1.8) and eq. (1.10) normalized appropriately. Note the zero crossing of $J_0(h)$ at $h \approx 2.40$: a radio engineer understand this as no transmitter power being wasted into the carrier frequency.

Finally, Fig. 1.7 shows the the approximations we found with the Bessel coefficients, taking into account the normalization eq. (1.11).

1.8 Demodulation

The explanation of demodulation that I like relies on the rotating frame.

Multiplying by $2e^{-j\omega_S t}$ stops the counter-clockwise rotating phasors [described by $e^{j(\omega_S t+\varphi_S)}$] and makes the clockwise one precesses at $-2\omega_S$:

$$V_S(t)2e^{-j\omega_S t} = A_S e^{j\varphi_S} + A_S e^{-j[2\omega_S t+\varphi_S]}.$$

Let us assume we can get rid of the fast rotating phasor by averaging/filtering, indicated by the symbol $\langle\ \rangle$: the averaged signal becomes

$$d(t) \equiv \left\langle V_S(t)2e^{-j\omega_S t}\right\rangle = A_S e^{j\varphi_S} \tag{1.12}$$

This is the demodulated signal.

To make the connection with the existing literature, where the real (or X or in-phase) and the imaginary (or Y or quadrature) components of the signal are shown, one notes that $e^{-j\omega_S t} \equiv \cos(\omega_S t) - j\sin(\omega_S t)$, from which $d(t) \equiv X + jY$ where

$$\begin{aligned} X &= \left\langle \mathrm{Re}\left\{V_S(t)2e^{-j\omega_S t}\right\}\right\rangle = \langle 2V_S(t)\cos(\omega_S t)\rangle = A_S\cos\varphi_S \\ Y &= \left\langle \mathrm{Im}\left\{V_S(t)2e^{-j\omega_S t}\right\}\right\rangle = \langle -2V_S(t)\sin(\omega_S t)\rangle = -A_S\sin\varphi_S. \end{aligned} \tag{1.13}$$

This is equivalent to multipying the input signal by cos and $-\sin$ of the reference signal.

Demodulation consists in two steps just enounced:

- multiplication (or mixing) of the input signal by $e^{-j\omega_S t}$ and
- filtering of the mixed signal: the filters should remove at least the 2ω component. (The type and implementation of the filters is not the subject of this note.)

1.8.1 Demodulation of the Signal: a Pictorial Representation

I restate the demodulation process using a pictorial representation. Let us start with the input signal eq. (1.1) and represent it as the sum of two phasors with equal length $\frac{A_S}{2}$ as before. Simple trigonometry is sufficient to show that the projection on the x–axis is $2 \cdot \frac{A_S}{2} \cos(\omega_S t)$ and has a null projection on the y–axis, see Fig. 1.8a) and b).

Figure 1.8: a) Decomposition of the input signal as two counter-rotating phasors of length $\frac{A_S}{2}$. b) The sum of the two phasors is always along the x–axis. c) In the rotating frame, one phasor is at rest and the other rotating at $2\omega_S$.

Imagine now to stand at the origin and to rotate counter-clockwise at the same speed ω_S: you will see one phasor at rest, at an angle φ_S with your forward direction (φ_S is the angle at time $t = 0$ of the input signal), while the other is rotating clockwise at an angular speed of $2\omega_S$ or to be more precise the angle between you and the second phasor is the time-dependent quantity $-2\omega_S t + \varphi_S$, Fig. 1.8c.

After averaging the total signal, only the steady phasor remains: the steady phasor has projection on the x–axis equal to $\frac{A_S}{2} \cos\varphi_S$ and on the y–axis equal to $\frac{A_S}{2} \sin\varphi_S$, the same result obtained in eq.(1.13)!

The factor $\frac{1}{2}$ comes from the decomposition of the real signal eq. (1.1) into the two vectors each of amplitude $\frac{A_S}{2}$, of which one averages out in the rotating frame. Therefore, since one phasor only is detected, it is customary to account for the missing amplitude in lock-in detection by displaying the X and Y components multiplied by 2 or $\sqrt{2}$ for peak or RMS amplitude respectively.

1.8.2 The Explanation (and Some Easy Math)

The rotating frame picture is equivalent to the mathematical approach: here is why.

Complex exponents are a very practical way of dealing with rotations in a 2D plane: given a vector (phasor) $Re^{j\theta}$ with length R and angle θ with the

x–axis, the phasor rotated by an angle ϕ is $Re^{j(\theta+\phi)} = Re^{j\theta} \cdot e^{j\phi}$, that is the product of the original phasor with the "rotation operator by an angle ϕ". The angle ϕ can also be a time-dependent angle, for instance $\phi(t) = \omega_S t$, in which case the phasor rotates at a constant speed.

I can equally well think of $e^{j\phi}$ as a rotation of the observer: if the phasor in the previous example appears at an angle $\theta + \phi$, it is because the observer has rotated by an angle $-\phi$, in the *clockwise* direction while the phasor remained fixed. So a rotation of the observer reference frame by an angle ϕ in the counter-clockwise direction multiplies by $e^{-j\phi}$, with the minus sign.

The change to a rotating frame of reference I mentioned above, with an angle $\phi(t) = \omega_R t$, corresponds to multiplying the input signal (each one of the two phasors) by $e^{-j\omega_R t}$. But $e^{-j\omega_R t}$ is just $\cos\omega_R t - j\sin\omega_R t$! This is why the cos and $-\sin$ multiplications to obtain the X (in-phase, the real part of the product) and the Y (the quadrature, the imaginary part) components.

As a side note: The Fourier transformation is a demodulation with infinitely long averaging

$$F(\omega_R) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-j\omega_R t}\mathrm{d}t.$$

Think of $f(t)$ as the sum of all components at different frequencies ω.[5] The recipe for the Fourier transform has two steps: first, rotate the reference frame at the angular speed ω_R, $f(t)e^{-j\omega_R t}$. Now, only the ω_R component is a steady phasor in this new rotating frame of reference. Second, take an infinitely long average, the time integration from $-\infty$ to $+\infty$ so that all the moving phasors having frequencies different from ω_R average out.

1.8.3 Demodulation at a Different Frequency

The pictorial representation is as correct as the mathematical approach and by no means an approximation of it. We will now only make use of the pictorial representation and predict what happens to the demodulated signal when the reference/demodulation frequency and the signal frequency differ, $\omega_R \neq \omega_S$; we then rigorously derive it mathematically and prove our intuition correct.

Let us assume that the signal's frequency is higher than the reference's, $\omega_S > \omega_R$: in the rotating frame, the slow moving phasor will appear moving counter-clockwise at a speed $\omega_S - \omega_R$ while as before, the fast one rotating at $-(\omega_S + \omega_R)$ is averaged out.

This is our result: the phase changes at a rate of $\omega_S - \omega_R$ to which we need to add the phase φ_S at $t = 0$, to give $(\omega_S - \omega_R)t + \varphi_S$. Using now the mathematical approach, we want to check the validity of the result: the multiplication by the reference signal $e^{-j\omega_R t}$ and the subsequent averaging of the 2ω component gives

$$\left\langle V_S(t)2e^{-j\omega_R t}\right\rangle = \left\langle \frac{A}{2}\left[e^{j(\omega_S t+\varphi)} + e^{-j(\omega_S t+\varphi_S)}\right]2e^{-j\omega_R t}\right\rangle = Ae^{j[(\omega_S-\omega_R)t+\varphi_S]}$$

[5] Never mind that I explain the Fourier transform in terms of the Fourier transform: this is the proof by recursion.

a counterclockwise rotating phasor at frequency $\omega_S - \omega_R$: this is exactly the result we obtained in the rotating frame.